Quantum mechanical photon-count formula derived by entangled state representation

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By introducing the thermo entangled state representation, we derived four new photocount distribution formulas for a given density operator of light field. It is shown that these new formulas, which is convenient to calculate the photocount , can be expressed as such integrations over Laguree-Gaussian function with characteristic function, Wigner function, Q-function, and P-function, respectively.

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In quantum optics photon counting is important for judging the nonclassical features of light field, most measurements of the electromagnetic field are based on the absorption of photons via the photoelectric effect. This is true not only for used insofar as photodiodes, photomultipliers, etc., but also for such homely devices as the photographic plate and the eye. So the problem of photo-electric detection attracts an increasing attention of many physicists and scientists. Expressions for the detection probability have been presented in many works [1, 2]. The quantum mechanical photon counting distribution formula was first derived by Kelley and Kleiner [3]. As shown in Refs. [3–5] for the single radiation mode, the probability distribution $\mathfrak{p}\left(m,T\right)$ of registering m photoelectrons in the time interval T is given by

$$\mathfrak{p}\left(m,T\right)=\mathtt{Tr}\left\{\rho\colon\frac{\left(\zeta a^{\dagger}a\right)^{m}}{m!}e^{-\zeta a^{\dagger}a}\colon\right\}, \tag{1}$$

where $\zeta\propto T$ is called the *quantum efficiency* (a measure) of the detector, and $:\ :$ denotes normal ordering. ρ is a single-mode density operator of the light field concerned. The aim of this Letter is to derive some other quantum mechanical photon-count formula by introducing the thermal entangled state representation and convert the calculations of Wigner function (WF) and the characteristic function of density operator to an overlap between "two pure"states in a two-mode enlarged Fock space, so that it is convenient to calculate the photocount when a light field's density operator is given. In addition, this new method seems concise and easy to be accepted by readers.

Recall that the thermal entangled state representation (TESR) is constructed in the doubled Fock space [6, 7] based on Umezawa-Takahash thermo field dynamics (TFD) [8–10], i.e.,

$$\begin{aligned}\left|\eta\right\rangle &= \exp\left[-\frac{1}{2}|\eta|^{2}+\eta a^{\dagger}-\eta^{*}\tilde{a}^{\dagger}+a^{\dagger}\tilde{a}^{\dagger}\right]\left|0,\tilde{0}\right\rangle\\ &= D\left(\eta\right)\left|\eta=0\right\rangle,\end{aligned} \tag{2}$$

$$\begin{aligned}\left|\xi\right\rangle &= \exp\left[-\frac{1}{2}|\xi|^{2}+\xi a^{\dagger}+\xi^{*}\tilde{a}^{\dagger}-a^{\dagger}\tilde{a}^{\dagger}\right]\left|0,\tilde{0}\right\rangle\\ &= D\left(\xi\right)\left|\xi=0\right\rangle,\end{aligned} \tag{3}$$

where the state vector $\left|\xi\right\rangle$ is conjugate to the state $\left|\eta\right\rangle$, $D\left(\eta\right)=e^{\eta a^{\dagger}-\eta^{*}a}$ is a displacement operator, and $\tilde{a}^{\dagger}$ is a fictitious mode accompanying the real photon creation operator $a^{\dagger}$, $\left|0,\tilde{0}\right\rangle=\left|0\right\rangle\left|\tilde{0}\right\rangle$, and $\left|\tilde{0}\right\rangle$ is annihilated by $\tilde{a}$ with the relations $\left[\tilde{a},\tilde{a}^{\dagger}\right]=1$ and $\left[a,\tilde{a}^{\dagger}\right]=0$. It is easily seen that $\left|\eta=0\right\rangle$ and $\left|\xi=0\right\rangle$ have the properties,

$$\left|I\right\rangle \equiv \left|\eta=0\right\rangle=e^{a^{\dagger}\tilde{a}^{\dagger}}\left|0,\tilde{0}\right\rangle=\sum_{n=0}^{\infty}\left|n,\tilde{n}\right\rangle, \tag{4}$$

$$\left|\xi=0\right\rangle = \left(-1\right)^{a^{\dagger}a}\left|\eta=0\right\rangle, \tag{5}$$

where $\tilde{n}=n$, and $\tilde{n}$ denotes the number in the fictitious Hilbert space.

According to the TFD and Eq.(4), we can reform the probability distribution $\mathfrak{p}\left(m,T\right)$ as

$$\begin{aligned}\mathfrak{p}\left(m,T\right) &= \sum_{n=0}^{\infty}\left\langle n\right|\rho\colon\frac{\left(\zeta a^{\dagger}a\right)^{m}}{m!}e^{-\zeta a^{\dagger}a}\colon\left|n\right\rangle\\ &= \sum_{n,l=0}^{\infty}\left\langle n,\tilde{n}\right|\rho\colon\frac{\left(\zeta a^{\dagger}a\right)^{m}}{m!}e^{-\zeta a^{\dagger}a}\colon\left|l,\tilde{l}\right\rangle\\ &= \frac{\zeta^{m}}{m!}\left\langle\rho\right|a^{\dagger m}\left(1-\zeta\right)^{a^{\dagger}a}a^{m}\left|I\right\rangle,\end{aligned} \tag{6}$$

where in the last step, we have used the operator identity: $\exp\left(\lambda a^{\dagger}a\right)=\colon\exp\left[\left(e^{\lambda}-1\right)a^{\dagger}a\right]\colon$. Note that the density operators $\rho(a^{\dagger},a)$ are defined in the real space which are commutative with operators $(\tilde{a}^{\dagger},\tilde{a})$ in the tilde space with $\left|\rho\right\rangle\equiv\rho\left|I\right\rangle$, as well as $\left\langle\tilde{n}\right|\left.\tilde{l}\right\rangle=\delta_{n,l}$ $(n=\tilde{n},l=\tilde{l})$. By using $a^{m}\left|l\right\rangle=\sqrt{l!/(l-m)!}\left|l-m\right\rangle, a^{\dagger m}\left|l\right\rangle=$

$\sqrt{(l+m)!/l!}\left|l+m\right\rangle$, Eq.(6) becomes

$$\begin{aligned}\mathfrak{p}(m,T) &= \frac{\zeta^m}{m!}\langle\rho|\sum_{l=0}^{\infty}\frac{(l+m)!}{l!}(1-\zeta)^l\left|l+m,\widetilde{l+m}\right\rangle \\ &= \zeta^m\langle\rho|\sum_{l=0}^{\infty}\frac{\left[(1-\zeta)a^\dagger\tilde{a}^\dagger\right]^l}{l!}|m,\tilde{m}\rangle \\ &= \zeta^m\langle\rho|e^{(1-\zeta)a^\dagger\tilde{a}^\dagger}|m,\tilde{m}\rangle. \end{aligned} \tag{7}$$

In order to derive four new formulas for $\mathfrak{p}(m,T)$, we first bridge the relation between the characteristic function (CF) and the entangled state representation $\langle\eta|$. Similarly to Eqs.(6), after using the TFD theory, the CF of density operator ρ, $\chi_S(\lambda,\lambda^*)=\mathtt{tr}\left(\rho e^{\lambda a^\dagger-\lambda^* a}\right)$, can be calculated as

$$\begin{aligned}\chi_S(\lambda,\lambda^*) &= \sum_{m,n}^{\infty}\langle n,\tilde{n}|\rho e^{\lambda a^\dagger-\lambda^* a}|m,\tilde{m}\rangle \\ &= \langle\rho|D(\lambda)|\eta=0\rangle \\ &= \langle\rho|\eta=\lambda\rangle, \end{aligned} \tag{8}$$

which is the CF formula in thermo entangled state representation, with which the characteristic function of density operator is simplified as an overlap between two "pure states" in enlarged Fock space, rather than using ensemble average in the system-mode space. Thus we can then simplify the calculation of $\chi_S(\lambda,\lambda^*)$ by virtue of some important properties of the entangled state representation $\langle\eta|$.

Using the expression of $\langle\eta|$ in Fock space, i.e.,

$$\langle\eta|=\langle 0,\tilde{0}|\sum_{m,n=0}^{\infty}i^{m+n}\frac{a^m\tilde{a}^n}{m!n!}H_{m,n}(-i\eta^*,i\eta)e^{-|\eta|^2/2}, \tag{9}$$

where $H_{m,n}(\xi^*,\xi)$ is the two-variable Hermite polynomials [11, 12], one finds

$$\langle\eta|m,\tilde{n}\rangle=i^{m+n}H_{m,n}(-i\eta^*,i\eta)e^{-|\eta|^2/2}/\sqrt{m!n!}, \tag{10}$$

which leads to

$$\begin{aligned}&\langle\eta|e^{(1-\zeta)a^\dagger\tilde{a}^\dagger}|m,\tilde{m}\rangle \\ &= \sum_{n=0}^{\infty}\frac{(1-\zeta)^n}{n!}\frac{(m+n)!}{m!}\left\langle\eta\,\middle|m+n,\widetilde{m+n}\right\rangle \\ &= \frac{(-1)^m e^{-|\eta|^2/2}}{m!}\sum_{n=0}^{\infty}\frac{(\zeta-1)^n}{n!}H_{m+n,m+n}(-i\eta^*,i\eta) \\ &= \frac{1}{\zeta^{m+1}}e^{-\frac{2-\zeta}{2\zeta}|\eta|^2}L_m\left(\frac{1}{\zeta}|\eta|^2\right), \end{aligned} \tag{11}$$

where in the last step, we have used the formula [13],

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}H_{m+l,n+l}(x,y) \\ &= \frac{e^{\frac{\alpha xy}{\alpha+1}}}{(\alpha+1)^{(m+n+2)/2}}H_{m,n}\left(\frac{x}{\sqrt{\alpha+1}},\frac{y}{\sqrt{\alpha+1}}\right), \end{aligned} \tag{12}$$

and the relation between two-variable Hermite polynomials and Laguree polynomials,

$$L_m(xy)=\frac{(-1)^m}{m!}H_{m,m}(x,y). \tag{13}$$

Further inserting the completeness relation of $\langle\eta|$, i.e., $\int\frac{1}{\pi}\mathtt{d}^2\eta\,|\eta\rangle\langle\eta|=1$ (it can be proved by using the normally ordered form of vacuum projector $|0,\tilde{0}\rangle\langle 0,\tilde{0}|=:\exp\left(-a^\dagger a-\tilde{a}^\dagger\tilde{a}\right):$ and the technique of integration within an ordered product (IWOP) of operators [14–16]), into Eq.(7), we can rewrite it as

$$\mathfrak{p}(m,T)=\frac{1}{\zeta}\int\frac{\mathtt{d}^2\lambda}{\pi}e^{-\frac{2-\zeta}{2\zeta}|\lambda|^2}L_m\left(\frac{1}{\zeta}|\lambda|^2\right)\chi_S(\lambda,\lambda^*), \tag{14}$$

which is just a new relation about the CF and the photon-count distribution. When the characteristic function $\chi_S(\lambda,\lambda^*)$ of density operator for Wigner-Weyl form is known, the photocount distribution can be calculated by using Eq.(14).

For instance, we first consider the single-mode coherent states $|\beta\rangle$, whose CF reads

$$\chi_{\mathrm{coh}}(\lambda,\lambda^*)=\exp\left[-\frac{1}{2}|\lambda|^2+\lambda\beta^*-\lambda^*\beta\right], \tag{15}$$

substituting it into Eq.(14) yields

$$\begin{aligned}\mathfrak{p}(m,T) &= \int\frac{\mathtt{d}^2\lambda}{\pi\zeta}e^{-\frac{1}{\zeta}|\lambda|^2+\lambda\beta^*-\lambda^*\beta}L_m\left(\frac{1}{\zeta}|\lambda|^2\right) \\ &= \frac{(\zeta\bar{n})^m}{m!}e^{-\zeta\bar{n}},\ (\bar{n}=\langle\beta|a^\dagger a|\beta\rangle=|\beta|^2), \end{aligned} \tag{16}$$

where we use the limiting expression $\lim_{x\to 0}x^m L_m(-|\alpha|^2/x)=\frac{1}{m!}|\alpha|^{2m}$ and the following integrational formula (see Appendix B),

$$\begin{aligned}&\int\frac{d^2\alpha}{\pi}e^{-B|\alpha|^2+C\alpha-C^*\alpha^*}L_m\left\{A|\alpha|^2\right\} \\ &= \frac{(B-A)^m}{B^{m+1}}e^{\frac{-CC^*}{B}}L_m\left(\frac{ACC^*/B}{A-B}\right). \end{aligned} \tag{17}$$

Eq.(16) is the Poisson distribution coinciding with the result in Refs. [4, 5].

As another example, we consider the single-mode squeezed vacuum state, $\exp\left[r\left(a^{\dagger 2}-a^2\right)/2\right]|0\rangle$, whose CF reads

$$\chi_{sq}(\lambda,\lambda^*)=\exp\left[-\frac{1}{2}|\lambda|^2\cosh 2r+\frac{1}{4}\left(\lambda^2+\lambda^{*2}\right)\sinh 2r\right], \tag{18}$$

substituting Eq.(18) into (14), we have (Appendix C)

$$\begin{aligned}\mathfrak{p}(m,T) &= \frac{\xi^m\operatorname{sech}r\tanh^m r}{\left(V^2-1\right)^{m/2}\left(1-V^2\right)^{1/2}}P_m\left(\frac{V}{\sqrt{V^2-1}}\right), \\ (V &= (1-\xi)\tanh r), \end{aligned} \tag{19}$$

which $P_m(x)$ is the Legendre polynomial and Eq.(19) is a new result.

Next, we derive other three new formula. Notice that the characteristic function $\chi_S(\lambda,\lambda^*)$ is related to the Wigner function, Q-function and P-representation by the following Fourier transforms,

$$\chi_S(\lambda,\lambda^*) = \int e^{\lambda\alpha^*-\lambda^*\alpha}W(\alpha)\,d^2\alpha, \tag{20}$$

$$\chi_S(\lambda,\lambda^*) = e^{\frac{|\lambda|^2}{2}}\int e^{\lambda\alpha^*-\lambda^*\alpha}Q(\alpha)\,d^2\alpha, \tag{21}$$

$$\chi_S(\lambda,\lambda^*) = e^{-\frac{|\lambda|^2}{2}}\int e^{\lambda\alpha^*-\lambda^*\alpha}P(\alpha)\,d^2\alpha, \tag{22}$$

respectively, thus substituting Eqs.(20)-(22) into (14) we can directly obtain

$$\mathfrak{p}(m,T) = \frac{2(-\zeta)^m}{(2-\zeta)^{m+1}}\int d^2\alpha e^{-\frac{2\zeta|\alpha|^2}{2-\zeta}}L_m\left\{\frac{4|\alpha|^2}{2-\zeta}\right\}W(\alpha), \tag{23}$$

$$\mathfrak{p}(m,T) = \frac{(-\zeta)^m}{(1-\zeta)^{m+1}}\int d^2\alpha e^{\frac{-\zeta|\alpha|^2}{1-\zeta}}L_m\left\{\frac{|\alpha|^2}{1-\zeta}\right\}Q(\alpha), \tag{24}$$

$$\mathfrak{p}(m,T) = \frac{\zeta^m}{m!}\int d^2\alpha\,|\alpha|^{2m}e^{-\zeta|\alpha|^2}P(\alpha), \tag{25}$$

where $W(\alpha)=2\mathtt{tr}(\rho\Delta(\alpha,\alpha^*)),Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$, and the integrational formula (17) is used. Eqs.(23)-(25) are the new formula for evaluating photon count distribution. Therefore, once one of these distributions of ρ is known, the photocount distribution can be calculated by using Eq.(23)-(25), which involve the Wigner function, Q-function, and P-representation of ρ, respectively. To confirm their correctness, we still consider the coherent light field $|\beta\rangle\langle\beta|$, its Wigner function, Q-function and P-function are given by $W(\alpha)=\frac{2}{\pi}e^{-2|\beta-\alpha|^2}$, $P(\alpha)=\delta^{(2)}(\beta-\alpha)$, and $Q(\alpha)=\frac{1}{\pi}e^{-|\beta-\alpha|^2}$, respectively, then according to (23)-(25) and using (17) and the above limiting expression $\lim_{x\to0}x^mL_m(-|\alpha|^2/x)=\frac{1}{m!}|\alpha|^{2m}$, one can draw the same result as Eq.(16).

At last, we should mention that using Eqs. (2)-(5) it is shown that the Wigner function of a mixed state ρ, $W_\rho(\alpha,\alpha^*)\equiv2\mathtt{tr}(\Delta(\alpha,\alpha^*)\rho)$, where $\Delta(\alpha,\alpha^*)$ is the single-mode Wigner operator [17, 18], whose explicit normally ordered form is [19]

$$\Delta(\alpha,\alpha^*)=\frac{1}{\pi}:e^{-2\left(a^\dagger-\alpha^*\right)(a-\alpha)}:=\frac{1}{\pi}D(2\alpha)(-1)^{a^\dagger a}, \tag{26}$$

which can also be converted to a overlap between two "pure state"in the enlarged Fock space,

$$\begin{aligned}W_\rho(\alpha,\alpha^*) &= \sum_{m,n}^{\infty}\langle n,\tilde{n}|\Delta(\alpha,\alpha^*)\rho|m,\tilde{m}\rangle\\ &= \frac{1}{\pi}\langle\eta=0|D(2\alpha)(-1)^{a^\dagger a}|\rho\rangle\\ &= \frac{1}{\pi}\langle\eta=-2\alpha|(-1)^{a^\dagger a}|\rho\rangle\\ &= \frac{1}{\pi}\langle\xi=2\alpha|\rho\rangle,\end{aligned} \tag{27}$$

which is the Wigner function formula in thermo entangled state representation, with which the Wigner function of density operator is simplified as an overlap between two "pure states" in enlarged Fock space. Employing its completeness, i.e., $\int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1$, one can derive these above new formula. In addition, the expression in Eq.(27) can also examine the evolution of Wigner function of density operator interacting with the environments [20].

In summary, based on Umezawa-Takahash thermo field dynamics theory, after introducing the thermo entangled state representation, we converted the calculation of CF to an overlap between two "pure states" in enlarged Fock space. Then we bridge the relation between the characteristic function and the photo-count distribution. Once the CF of density operator for Wigner-Weyl form is known, the photocount distribution can be calculated conveniently. Using the Fourier transform relation between the CF and the distribution functions, we further derive other three new formula so as to be convenient for calculating photo-count distribution by using these formulas.

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Appendix A: Derivation of sum-formula in Eq.(12)

Using the integration of two-variable Hermite polynomials,

$$H_{m,n}(\xi,\eta)=(-1)^ne^{\xi\eta}\int\frac{d^2z}{\pi}z^nz^{*m}e^{-|z|^2+\xi z-\eta z^*}, \tag{A1}$$

we have

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}H_{m+l,n+l}(x,y)\\ &=\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}(-1)^{n+l}e^{xy}\int\frac{d^2z}{\pi}z^{n+l}z^{*m+l}e^{-|z|^2+xz-yz^*}\\ &=e^{xy}(-1)^n\int\frac{d^2z}{\pi}z^nz^{*m}e^{-(\alpha+1)|z|^2+xz-yz^*}.\end{aligned} \tag{A2}$$

Then making scale transform and using Eq.(A1) again, Eq.(A2) can be put into the following form

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}H_{m+l,n+l}(x,y)\\ &=\frac{(-1)^ne^{xy}}{(\alpha+1)^{(m+n+2)/2}}\int\frac{d^2z}{\pi}z^nz^{*m}e^{-|z|^2+\frac{xz}{\sqrt{\alpha+1}}-\frac{yz^*}{\sqrt{\alpha+1}}}\\ &=\text{Right hand side of Eq.(12).}\end{aligned} \tag{A3}$$

Appendix B: Derivation of integration-formula in Eq.(17)

Using Eq.(13) and the generating function of the two-variable Hermite polynomials,

$$\left.\frac{\partial^{m+n}}{\partial\tau^m\partial\upsilon^n}e^{-A\tau\upsilon+B\tau+C\upsilon}\right|_{\tau=\upsilon=0}=\left(\sqrt{A}\right)^{m+n}H_{m,n}\left(\frac{B}{\sqrt{A}},\frac{C}{\sqrt{A}}\right), \tag{B1}$$

we find

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}L_m\left\{A^2\left|\alpha\right|^2\right\}e^{-B^2|\alpha|^2+C\alpha+C^*\alpha^*}&=\int\frac{d^2\alpha}{\pi}\frac{(-1)^m}{m!}H_{m,m}\left\{A\alpha,A\alpha^*\right\}e^{-B^2|\alpha|^2+C\alpha+C^*\alpha^*}\\&=\frac{(-1)^m}{m!}\frac{\partial^{2m}}{\partial t^m\partial t'^m}e^{-tt'}\int\frac{d^2\alpha}{\pi}\left.e^{-B^2|\alpha|^2+(C+At)\alpha+\left(C^*+At'\right)\alpha^*}\right|_{t=t'=0}\\&=\frac{(-1)^m}{m!}\frac{\left(B^2-A^2\right)^m}{B^{2(m+1)}e^{-CC^*/B^2}}\frac{\partial^{2m}}{\partial t^m\partial\tau^m}\left.e^{-t\tau+\frac{AC/B}{\sqrt{B^2-A^2}}\tau+\frac{AC^*/B}{\sqrt{B^2-A^2}}t}\right|_{t=\tau=0},\end{aligned} \tag{B2}$$

where we have used the formula

$$\int\frac{d^2z}{\pi}e^{\zeta|z|^2+\xi z+\eta z^*}=-\frac{1}{\zeta}e^{-\frac{\xi\eta}{\zeta}},\mathrm{Re}\left(\zeta\right)<0 \tag{B3}$$

Using Eqs.(B1) and (13) again, one can get the integration-formula in Eq.(17).

Appendix C: Derivation of the result in Eq.(19)

In order to obtain Eq.(19), we first derive a new integral formula,

$$I\equiv\int\frac{d^2\lambda}{\pi}L_m\left\{A\left|\lambda\right|^2\right\}e^{-B|\lambda|^2+C\lambda^2+C\lambda^{*2}}. \tag{C1}$$

Using Eqs.(13) and (B1), Eq.(C1) can be put into the form

$$\begin{aligned}I&=\frac{(-1)^m}{m!}\int\frac{d^2\lambda}{\pi}H_{m,m}\left(\sqrt{A}\lambda,\sqrt{A}\lambda^*\right)e^{-B|\lambda|^2+C\lambda^2+C\lambda^{*2}}\\&=\frac{(-1)^m}{m!}\frac{\partial^{2m}}{\partial t^m\partial\tau^m}e^{-\tau t}\int\frac{d^2\lambda}{\pi}\left.e^{-B|\lambda|^2+t\sqrt{A}\lambda+\tau\sqrt{A}\lambda^*+C\lambda^2+C\lambda^{*2}}\right|_{t=\tau=0}\\&=\frac{(-1)^m}{m!\sqrt{B^2-4C^2}}\frac{\partial^{2m}}{\partial t^m\partial\tau^m}\exp\left[-\frac{B^2-4C^2-BA}{B^2-4C^2}\tau t+\frac{CA\left(\tau^2+t^2\right)}{B^2-4C^2}\right]_{t=\tau=0},\end{aligned} \tag{C2}$$

where in the last step, we used the formula [21]

$$\begin{aligned}&\int\frac{d^2z}{\pi}\exp\left(\zeta\left|z\right|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)\\&=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\left[\frac{-\zeta\xi\eta+\xi^2g+\eta^2f}{\zeta^2-4fg}\right],\end{aligned} \tag{C3}$$

whose convergent condition is $\mathrm{Re}(\zeta\pm f\pm g)<0$, $\mathrm{Re}[(\zeta^2-4fg)/(\zeta\pm f\pm g)]<0$.

Expanding the exponential item involved in Eq.(C2), we see

$$\begin{aligned}I&=\frac{(-1)^m}{m!\sqrt{B^2-4C^2}}\sum_{n,l,k=0}^{\infty}\frac{(-1)^k}{n!l!k!}\frac{\left(B^2-4C^2-BA\right)^k}{\left(B^2-4C^2\right)^{k+n+l}/\left(CA\right)^{n+l}}\frac{\partial^{2m}}{\partial t^m\partial\tau^m}\left.\tau^{2n+k}t^{2l+k}\right|_{t=\tau=0}\\&=\frac{\left(B^2-4C^2-BA\right)^m}{\left(B^2-4C^2\right)^{m+1/2}}\sum_{l=0}^{[m/2]}\frac{m!}{2^{2l}l!l!\left(m-2l\right)!}\left(\frac{1}{y}\right)^{2l},\end{aligned} \tag{C4}$$

where

$$y=\frac{B^2-4C^2-AB}{2AC}. \tag{C5}$$

Recalling that newly found expression of Lagendre polynomial (it is equivalence to the well-known Legendre polynomial's expression [22]),

$$x^m\sum_{l=0}^{[m/2]}\frac{m!}{2^{2l}l!l!\,(m-2l)!}\left(1-\frac{1}{x^2}\right)^l=P_m\left(x\right), \tag{C6}$$

the compact form for I is written as

$$I=\frac{\left((A-B)^2-4C^2\right)^{m/2}}{\left(B^2-4C^2\right)^{(m+1)/2}}P_m\left(\frac{y}{\sqrt{y^2-1}}\right), \tag{C7}$$

which is a new integration formula.

Substituting Eq.(18) into (14) we have

$$\mathfrak{p}\left(m,T\right)=\frac{1}{\zeta}I', \tag{C8}$$

where I' shown in Eq.(C7) characteristic of

$$A=\frac{1}{\zeta},B=\frac{1}{\zeta}+\sinh^2 r,C=\frac{1}{4}\sinh 2r, \tag{C9}$$

which leads to

$$y=\left(1-\zeta\right)\tanh r, \tag{C10}$$

$$A-B=\left(A-B\right)^2-4C^2=-\sinh^2 r, \tag{C11}$$

$$B^2-4C^2=\frac{1}{\zeta^2}\left[\left(2-\zeta\right)\zeta\sinh^2 r+1\right], \tag{C12}$$

and

$$\begin{aligned}&\frac{\left((A-B)^2-4C^2\right)^{m/2}}{\left(B^2-4C^2\right)^{(m+1)/2}}\\&=\frac{\zeta^{m+1}\mathrm{sech}r\left(-\tanh^2 r\right)^{m/2}}{\left(\left(2-\zeta\right)\zeta\tanh^2 r+\mathrm{sech}^2 r\right)^{(m+1)/2}}\\&=\frac{\zeta^{m+1}\mathrm{sech}r\left(-\tanh^2 r\right)^{m/2}}{\left(1-y^2\right)^{(m+1)/2}},\end{aligned} \tag{C13}$$

so

$$\mathfrak{p}\left(m,T\right)=\frac{\zeta^m\mathrm{sech}r\tanh^m r}{\left(y^2-1\right)^{m/2}\left(1-y^2\right)^{1/2}}P_m\left(\frac{y}{\sqrt{y^2-1}}\right), \tag{C14}$$

which is the photon-count distribution of squeezed vacuum state.

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